ICERM Workshop

Lattice point counting and homogeneous dynamics

June 2020

Lattice point counting and homogeneous Dynamics

Homogeneous space $\Gamma\backslash G$

Γ - a lattice

(discrete s.g. $\mathrm{Vol}(\Gamma\backslash G)<\infty$

Dynamics: action of subgroup $H\leq G$ on

$\Gamma\backslash G \circlearrowleft H$

Lattice points on G (G/H)

$\longleftrightarrow$

orbits of H on $\Gamma\backslash G$

Counting $\Gamma\cap B_T$

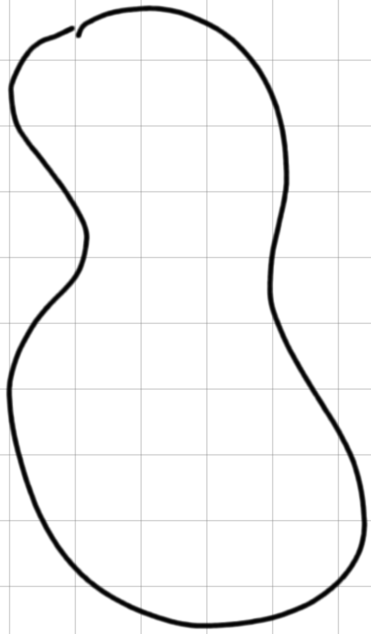
$B_T\subseteq G$

Today: $G = \mathbb{R}^2 \quad \Gamma = \mathbb{Z}^2$

$B_T \subseteq \mathbb{R}^2$ nice growing sets

count: $|\mathbb{Z}^2 \cap B_T|$

Geometric principle



B_T^+ - thickening of B_T

cover $m \in B_T$ by 1×1 square

$$\mathrm{Area}(B_T^-) \le |\mathbb{Z}^2 \cap B_T| \le \mathrm{Area}(B_T^+)$$

$$\mathrm{Area}(B_T^+ \setminus B_T^-) \approx |\partial B_T|$$

conclusion

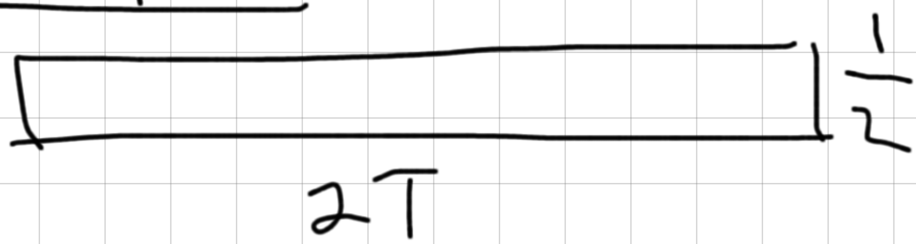
$$|\mathbb{Z}^2 \cap B_T| = \text{Area}(B_T) + O(|\partial B_T|)$$

Question: what if

$$\text{Area}(B_T) \approx |\partial B_T| \ ?$$

Examples:

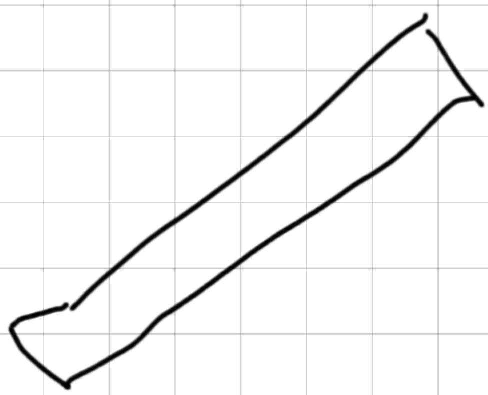
B_T



$$\text{Area}(B_T) = T \qquad |\partial B_T| = 4T+1$$

$$0 \leq |B_T \cap \mathbb{Z}^2| \leq 2T+1$$

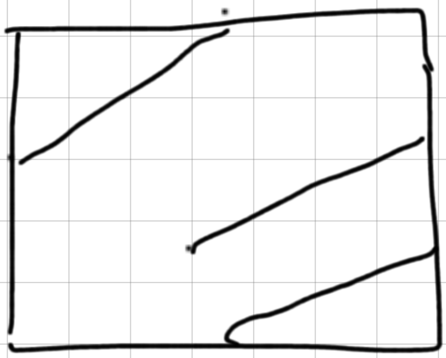
What if we rotate B_T



Dynamics on $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

Fix $x_0 \in \mathbb{T}$ $\underline{u} \in S^1$

$v_t = v_t(x_0, \underline{u}) = \underline{x}_0 + t\underline{u} \mod \mathbb{Z}^2$



__slope__: $\alpha = \frac{u_2}{u_1}$

if $\alpha \in \mathbb{Q}$, orbit is closed

for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

__claim__: The orbit

$\{v_t \mid |t| \leq T\}$ equidistributes

on $\mathbb{T}$ as $T \to \infty$

This means any of the following:

1) $\forall A \subseteq \mathbb{T}$ $\quad Area(\partial A) = 0$

$$\frac{\left|\{|t|<T \mid v_t \in A\}\right|}{2T} \to Area(A)$$

2) $\forall f \in L^2(\mathbb{T})$ $\quad f$ - cont.

$$\frac{1}{2T}\int_{-T}^{T} f(v_t)\,dt \to \int_{\mathbb{T}} f$$

3) $\forall m \in \mathbb{Z}^2 \setminus 0$

$$\frac{1}{2T}\int_{-T}^{T} e^{2\pi i m \cdot v_t}\,dt \to 0$$

For $v_t = x_0 + \underline{y} t$

slope $\alpha \notin \mathbb{Q} \Rightarrow$

$\underline{m} \cdot \underline{y} \neq 0 \quad \forall \underline{m} \neq 0$

$$\frac{1}{2T} \int_{-T}^{T} e^{2\pi i \underline{m} \cdot v_t} \, dt$$

$$= e^{2\pi i \underline{m} \cdot x_0} \frac{\sin(2\pi(\underline{m} \cdot \underline{y}) T)}{2\pi (\underline{m} \cdot \underline{y}) T}$$

$$\longrightarrow 0$$

<u>Note:</u> This can be made effective

Assume f smooth

α is not very well approximable $|p - q\alpha| \geq \frac{1}{q^c}$ for $c > 1$ for all but finite many q

Exe - for α not UWA

$$\frac{1}{2T}\int_{-T}^{T} f(v_t)\,dt = \int f + O\left(\frac{\|\nabla f\|}{T}\right)$$

Back to counting

Fix $\underline{u} \in S^1$ slope α

$\underline{v} \in S^1$ $\quad \underline{v}\cdot\underline{u} = 0$

$$B_T = \left\{ t\underline{u} + s\underline{v} \;\middle|\; \begin{array}{l} |t| \le T \\ |s| \le \frac{1}{4} \end{array} \right\}$$

Thm

$$\frac{\#(B_T \cap \mathbb{Z}^2)}{T} \longrightarrow 1$$

Pf. Fix $\delta > 0$ and

$U_\delta = \{ \underline{x} \mid \|\underline{x}\| < \delta \}$

φ_δ supported in U_δ

$$\int \varphi_\delta(x)\, dx = 1$$

$$\Phi_\delta(\underline{x}) = \sum_{\underline{m} \in \mathbb{Z}^2} \varphi_\delta(\underline{x} + \underline{m})$$

calculate

$$\int_{B_1} \Phi_\delta(\underline{x})\, d\underline{x}$$

$$\int_{B_T} \Phi(x)\,dx = \sum_{\underline{y}} \int_{\mathbb{R}^n} \chi_{B_T}(x)\, \varphi_j(x+y)\,dx$$

$$= \sum_{\underline{y}n} \int_{U_8} \chi_{B_T}(x+y)\, \varphi_j(x)\,dx$$

Let

$$B_T^{\pm} = \left\{ t\underline{u} + s\underline{v} \;\middle|\; \begin{array}{l} |t| \leq T \pm \delta \\ |s| \leq \frac{T}{4} \pm \delta \end{array} \right\}$$

$$|B_T^- \cap \mathbb{Z}^2| \leq \int_{B_T} \Phi(x)\,dx \leq |B_T^+ \cap \mathbb{Z}^2|$$

$$\int_{B_T^-} \Phi \leq |\mathbb{Z}^2 \cap B_T| \leq \int_{B_T^+} \Phi(x)\,dx$$

$$\int_{B_T} \Phi(x)\,dx$$

$$= \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-T}^{T} \Phi(su+tu)\,dt\,du$$

so

$$\frac{1}{T}\int_{B_T} \Phi(x)\,dx$$

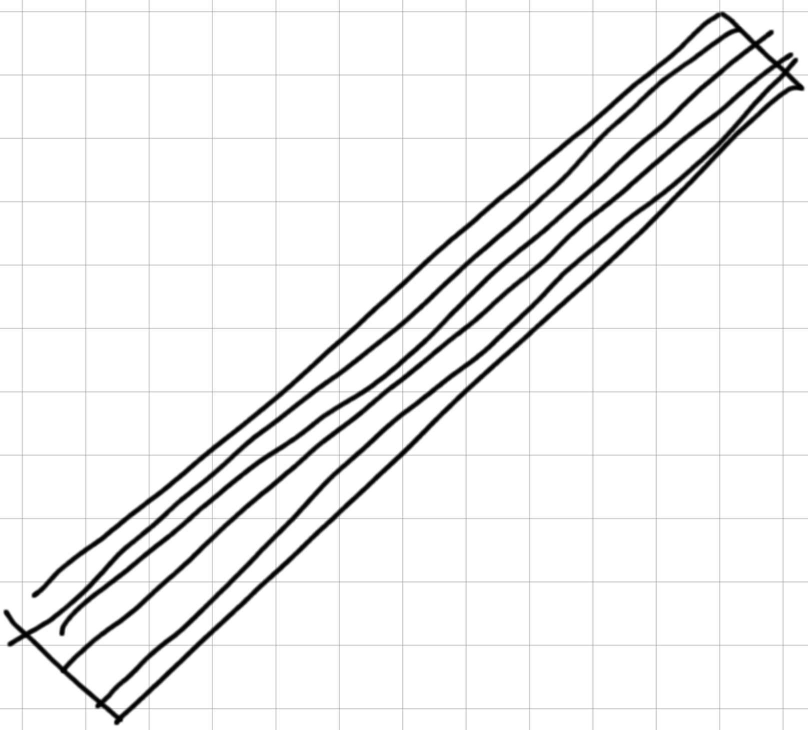
$$= 2\int_{-\frac{1}{4}}^{\frac{1}{4}} \left(\frac{1}{2T}\int_{-T}^{T} \Phi(v_t)\,dt\right) du$$

$$\xrightarrow{T\to\infty} \int_{\Pi} \Phi(x)\,dx = \int_{\mathbb{R}^2} \varphi_\delta = 1$$

Combining two estimates:

$$1-2\delta \le \lim_{T\to\infty} \frac{|B_T \cap \mathbb{Z}^2|}{T} \le 1+4\delta$$

Take $\delta \to 0$ to get result.



Note: This can be made effective for α is not V.W.A

$$|\mathbb{Z}^2 \cap B_T| = T + O(T^{3/4})$$

* This holds for a.e direction α

* Method works for more general sets growing in 1-direction